Is Arithmetic Synthetic? A Defense of Kant Against Frege's Logicistic Definition of Numbers

Here I will assess Kant and Frege's views on whether arithmetic is analytic or synthetic. I begin by presenting Kant's definitions of syntheticity and how it applies to arithmetic in the Critique of Pure Reason, and then proceed with Frege's framework of logicism in the Foundations of Arithmetic and his definitions of numbers through sets and equal numerosity. In the third section, I argue that Frege's definition of numbers using set theory is insufficient to prove arithmetic as analytic, for the concepts and sets under the logicistically defined subject and predicate do not fulfill the two necessary containment relations that are conditions to analyticity in reciprocity. Furthermore, I argue that Frege presupposed Kantian epistemology by invoking intuition. Overall, my essay presents a defense of Kant's assessment of arithmetic as synthetic a priori against the attack from Frege's logicism that holds arithmetic to be analytic.

Introduction:

Kant's *Critique of Pure Reason* (1998 CPR) sought to save science from the skeptic attacks on the possibility of knowledge of the empirical world. The empiricists consider the knowledge of science as concerning objects that are independent of our cognition and precedes our experience. Kant admits the existence of these transcendentally real objects; however, unlike the skeptics, his solution to defend science does not presuppose the standard of knowledge as conformed to the inexperienceable objects in themselves. Instead, Kant's epistemology presumes the standard of knowledge to be in our cognitive apparatus, where any knower possessing it can produce the same necessary and universal knowledge about the experienceable world, thereby defending science as true to all knowers subjected to these transcendental cognitive conditions.

But firstly, how could any necessary and universal knowledge be contaminated with the slightest connections to the experienceable at all? Since experience is contingent and arbitrary, how could knowers gather any a priori knowledge from them? For Kant, simple arithmetic judgments such as "7+5=12" exemplify such possibilities. In the Introduction of CPR, Kant categorizes arithmetic as synthetic a priori, where the truth of it is necessary and universal but is only confirmed with the contents received in experience such as counting. And on the basis that synthetic a priori knowledge exists, such as math, Kant then begins his endeavor to answer what cognitive conditions can allow such knowledge to be defensible against skepticism. However, is Kant correct that arithmetic is synthetic and a priori? Since the success of Kant's assessment of the knowability of true science rests on the existence of synthetic a priori knowledge, such as arithmetic, it is worth examining the soundness of his analysis of math.

Specifically, arithmetic is widely regarded as a priori truth, but many philosophers differ on whether it is analytic or synthetic. In this essay, I present a perspective by Gottlob Frege, who attempted to define numbers from logic such that arithmetic can be preserved as analytic. His approach is coined "logicism" and contributed greatly to the modern set theory. I argue that Frege's definition of numbers using set theory is insufficient to prove arithmetic as analytic, for the concepts and sets under the logicistically defined subject and predicate do not fulfill the two necessary containment relations that are conditions to the analyticity in reciprocity. Furthermore, I argue that Frege presupposed Kantian epistemology by invoking intuition. Overall, my essay presents a defense of Kant's assessment of arithmetic as synthetic a priori against the attack from Frege's logicism that holds arithmetic to be analytic.

Kant on A Priori Syntheticity and Arithmetic:

Kant famously claimed arithmetic to be a priori and synthetic. This section outlines his distinctions on these concepts and why arithmetic is categorized as such. Cognitions for Kant are categorized into either a priori or a posteriori. The former are independent of any experience and the latter are derived from experience. However, not all a priori cognitions are pure tautologies without any intermixes of empirical experiences, such as "A is A." Even though none are derived immediately from experience, some a priori cognitions are dependent on general rules "borrowed from experience" (CPR, B2). Kant illustrates this point using "all bodies are heavy", which cannot be known entirely a priori, and one become known of it with the experience of objects collapsing without proper support. Nevertheless, a priori cognitions are necessarily so through deduction and are strictly universal with no exceptions allowed. Therefore, a priori cognitions are necessary and universal, among which some could be known through experience, but all remain true regardless of the empirical.

Kant differentiates judgments into analytic and synthetic by three distinctions. 1) **By the relation between the concept of the subject S and the predicate P of a judgment** (CPR, A7). In analytic judgments, P can be thought of through the identity of S, where P belongs to S with its concept covertly contained in the concept of S. While in synthetic judgments, the concept P is connected to that of S but never contained by it. 2) **By the nature of the contents** (CPR, B11). It follows from the first distinction that analytic judgments must be explicative, where P does not add additional ideas to the concept of S and only clarifies S in terms of its implied components. Synthetic judgements are by opposition amplificative, where P extends our knowledge of S by adding something that cannot be definitionally extracted from S.3) **By their principles**. Analytic judgements are also a priori cognitions because they require no empirical experience to be derived and rely on only the principle of contradiction (CPR, B15). Conversely, synthetic judgements cannot be produced solely from analytic principles but demand a different principle which Kant later reveals as the unifying condition of manifolds in the intuitions of empirical objects (CPR, A158/B197). Overall, the first two definitions point to a subject-predicate relation in terms of their concepts, and the third one is related to the nature of their production.

In CPR, Kant's example for a synthetic judgement is the same one used to illustrate a priori cognitions that borrow rules from experience, "all bodies are heavy" (A8/B12). In opposition to the analytic judgement "all bodies are extended," where extension is analytically contained in the concept of bodies as spatial, heaviness is an external concept that amplifies the concept of bodies by adding weight to it. And this knowledge is only gathered under the unifying conditions of intuitions related to empirical sensations such as the collapse of objects.

In terms of math propositions, Kant has no doubt about their a priority. Because pure math is necessarily so when deducted and universally so with no outliers (CPR, B15). However, contrary to common beliefs of it being analytic due to its accordance with the principle of contradiction, Kant considers arithmetic equations such as "7+5=12" synthetic (CPR, B15). To begin understanding his argument, one must transform such proposition into a judgement in sentence form, where "the sum of number 7 and number 5 is number 12." In this judgement, the subject is "the sum of number 7 and number 5", and the predicate is "number 12". Then, one can assess this judgement using Kant's prior definitions of analyticity and syntheticity and draw the following observations: 1) The concept of "number 12" cannot be merely thought of analytically as contained in the concept of "the sum of number 7 and number 5" (CPR, B15), **which violates the analytic containment relation of the P concept under the S Concept**; 2) The predicate "number 12" also amplifies the subject concept of mere summation by introducing an external number concept as its result, **which violates the explicative property of analytic judgement**; 3) One must seek intuition of time to arrive at number 12 due to the limitations of the mere numerical concepts. Only by counting one's fingers in time consecutively can one get to the concept of 12 from the summation (CPR, B16), **which violates analytic judgements' sole dependence on analytic principles**. **Therefore, with these three violations of the necessary requirements of analytic judgements, Kant concluded arithmetic as a priori but synthetic.**

Very importantly, a priori syntheticity is not an inconsistent concept because certain a priori knowledge such as arithmetic can borrow from intuition to arrive at the judgment. And their syntheticity do not contradict their a priority because this knowledge is always true no matter who derives them or when and where they are derived.

Frege's Logicistic Approach to Numbers:

After Kant's bold claim of math as synthetic, many philosophers attempted to refute his argument for its sheer counter-intuitiveness. How could arithmetic, the a priori knowledge independent of the empirical, be anywhere near syntheticity? Nevertheless, if one follows Kant's definitions, a solution to save pure math from being synthetic seems to be available in the definition of arithmetic concepts. If we could define the concepts of numbers and their operations analytically, it would satisfy Kant's third definition of analyticity with all constituents of arithmetic deductible from sole analytic principles.

Frege approaches arithmetic precisely in this thinking, where he attempts to offer an analytical definition of numbers from axiomatic logic principles. Unfortunately, Kant made no similar effort to define numerical concepts in depth and only vaguely posited in CPR that "no matter how long I analyze my concept of such a possible sum, I will not find twelve in it" (B15). Therefore, if Frege's analytical definition of numbers can successfully give numerical concepts that satisfy Kant's first two definitions of analyticity, then Kant would be refuted. Unfortunately, I argue in the next section that Frege did not accomplish this task and is even foundationally in compliance with the third definition of syntheticity by implying intuition in his definitions.

Frege's account of arithmetic is based on the idea of logicism, which aims to reduce arithmetic to logic. His project to define numbers in *Foundations of Arithmetic* (1980 FA) begins by clarifying their nature as abstract objects because objectifying numbers is the foundation to have them recognized logically without appealing to intuition (FA, §45-§54). Frege first posits number words as objects in the context of language, such as in "number 12 is a positive integer". Secondly, number words never stand alone as complete predicates and are always attached to objects when describing a concept. For example, "this classroom has 12" does not complete the description of the subject until objects such as "students" are attached. And lastly, number words can always be separated from the predicate of an assertion as a distinct object. For example, "Jupiter has 4 moons" can be reparsed into "The number of Jupiter's moons is 4."

From the above rationales, Frege establishes that number words are abstract objects, and any statements of number objects form the content that extends a concept. In his own words, "the content of a statement of number is an assertion about a concept" (FA, §55-§57). For example, the assertion about the concept of "the number of Jupiter's moons" corresponds to the extended content of 80 moons. Therefore, Frege considers numbers as extensions of concepts whose content is composed of that many objects. This definition comes from the axiomatic truth that for any concept we could think of, there always exists corresponding extensions of it that form a set of objects under that concept. Since numbers are contained in statements about sets of objects by declaring their quantity, they can also be understood as the cardinality of these sets under the modern set theory that arose from Frege.

In order to illustrate Frege's definition of numbers formally in terms of set theory, certain concepts need to be clarified. Firstly, when numbers are indicating the quantity of objects in a set X, that number is called a cardinal which represents the cardinality of the set, expressed as n(X). For example, n({moons that belong to Jupiter})=80. The position of ordered objects in a set can also be indicated by numbers named ordinals. Secondly, the concept which an extension of objects fall under can be understood as a propositional function f(x), and these objects that fall under f(x) form a set of x and its corresponding value f(x), expressed as {x | f(x)}. Therefore, formally, any number n can be defined as the cardinality of a set X that is the extension to any concept f(x) that corresponds to sets of n members of any subject x.

However, without intuitions, the contents of any concept cannot be located. And without the empirical contents, there is no cardinality of sets that contains those contents. Therefore, Frege derives the number concept in terms of cardinality from Hume's principle, in which number words occur and carries meaning without referring to empirical objects (FA, §62-§63). This principle defines numerical identity in terms of sets, where the number of objects in a set F is equal to the number of objects in a set G if and only if there is a one-to-one correspondence (a.k.a. equinumerosity) between the objects of F and G. From here, Frege concludes that "the number which belongs to the concept F is the extension of the concept 'equal to the concept F" (FA, §68-§69). According to Frege, this definition makes sense of numbers as cardinalities of sets, deduced from analytic axioms, and do not explicitly contain intuition of any sets of objects. Therefore, he claims to have completed the task of reducing numbers to pure logical principles.

The Unfulfilled Subject-Predicate Reciprocity and the Implied Intuition in Logicism:

With Frege's definition of numbers that claimed to have deduced them from analytic principles, a test of arithmetic equations in terms of logicism is required to demonstrate its effectiveness in proving the analyticity of arithmetic.

According to Frege, any number N can be understood as the cardinality of any set X of objects x under the concept f(x) that contains n members, where n(X)=n. Then, 7, 5, and 12 can be defined as extensions to the concepts of all things under respective sets A, B, and C where n(A)=7, n(B)=5, and n(C)=12. Now, the subject concept "7+5" becomes the addition of two cardinals related to sets A and B, which requires their unionization. The union set of A and B simply entails the collection of all objects under them to make a set C′ that contains the subset A of 7 objects and the subset B of 5 objects expressed as A∪B= C′. Accordingly, the summation of the cardinalities of A and B becomes the cardinality of C′, expressed as n(A∪B) = n(C′). And finally, we can see that "7+5" can be represented as n(A∪B) or n(C′), which also gives 12, sustaining the a priori truth of "7+5=12" by having "n(C′)=n(C)". However, under this definition of numbers as cardinalities of sets, I argue that the new subject n(C′) and the new predicate n(C) do not satisfy the containment relation between the concepts of S and P under Kant's analyticity.

To begin my illustration of the unfulfilled concept-containment of P in S under Frege's logicism, let's suppose there is a world with only two categories of things, apples, and bananas. In this imagined world, 7 can only be the cardinality of the set A_1={7 apples} or the set A_2 ={7 bananas} where the two sets are in equinumerosity to each other. And the same analogy applies to numbers 5 and 12. Then, it follows that the S concept of "7+5" corresponds to the set C′/A∪B, where C′ has 4 or 2^2 forms of category combinations: $C_{11}/A_1 \cup B_1$ of {7 apples and 5 apples}, $C_{12}/A_1 \cup B_2$ of {7 apples and 5 bananas}, $C_{21}/A_2 \cup B_1$ of {7 bananas and 5 apples}, and $C_{22}/A_2 \cup B_2$ of {7 bananas and 5 bananas}. Also, the P concept of "12" corresponds to a set C that has 2 forms of category combinations: C_1 of {12 apples} and C_2 of {12 bananas}, where $C_1=C_{11}$ and $C_2=C_{22}$. In this case, from the set C′ that contains {C_{11} of 7 apples and 5 apples, C_{12} of 7 apples and 5 bananas, C_{21} of 7 bananas and 5 apples, C_{22} of 7 bananas and 5 bananas}, one cannot by concept arrive at the set C that contains just {C_1 of 12 apples, C_2 of 12 bananas}. To elaborate this point, a comparison to analytic judgements is needed.

In analytic judgements such as "all bodies are extended", the concept of P is definitionally contained in the concept of S where all bodies **must be** extended; but for synthetic concepts such as "some bodies are red" where redness is not necessarily contained in the definition of bodies, it only means that some bodies **could be** red. The arithmetic equation represented under logicistic definitions as n(C′)=n(C) clearly follows the case of synthetic judgements. Because the S of n(C′) where $C' \in \{ C_{11}, C_{12}, C_{21}, C_{22}\}$ does not entail in its concept that n(C′) must be n(C) where $C \in \{C_1, C_2\}$, only that **n(C′) could be n(C) when C′ is C_{11} of 12 apples or $C'=C_{22}$ of 12 bananas**. For all we know, n(C′) could also be n(C) when C′ is C_{12} or C_{21}, which are not the same as C at all. **Therefore, the P concept of n(C) is not contained in the S concept of n(C′) under logicistic definitions, which fails to fulfill Kant's first two requirements of analyticity where the concept of P must be contained in the concept of S.** And it is exactly this intuition of the objects themselves under the sets that are missing in this process of deriving C from C′, making n(C′)=n(C) bound to be synthetic relying on intuition.

The above analogy using apples and bananas can be explained formally as well. If there are m categories of things in this world where these categories form a set CAT, and any set N_m that corresponds to number concepts can only contain a single category of things associated with the ordinal m in the set

CAT, then for any arithmetic equation "x+y=z," the subject concept of "the addition of x and y" will always correspond to a set with m^2 forms of category combinations in terms of the union set C_{m1m2} of A_{m1} and B_{m2} with respective cardinalities x and y, and the set C that corresponds to the predicate concept of "z" will always have m forms of category combinations in terms of C_m with the cardinality z. To make sure the concept of P in terms of C is contained in the concept of S in terms of C′, it will always require intuitions of choosing the same m associated with A_{m1}, B_{m2}, and C_m. In the prior example, in the world where CAT= {apples, bananas}, I must rely on intuition to choose that A, B, and C all contain the same object such as apples, making m=m1=m2=1. **Only in this way, C′ and C contain the exact same sets of apples, and therefore to be the subject n(C′) must mean to be the predicate n(C). The intuition of choosing apples for all set definitions is crucial here, therefore, making the arithmetic judgement under logicistic definitions synthetic with its dependence on empirical objects.**

An objection against my argument might point out that the set C with m forms of C_m under P always forms a subset to the set C′ with m^2 forms of C_{m1m2} under S, therefore making the concept of n(C) contained in the concept of n(C′). However, this is the exact opposite of the set relations under S and P in an analytic judgement, and more importantly, **I argue that this objection obscures two different kinds of relations between S and P that forms reciprocity**.

In analytic judgements, the concept of P is contained in S., and the set under S is either a subset of the set under P or is in strict identity with the set under P. Either way, the set under S is contained in the set under P. For example, in judgements such as "all bachelors are unmarried men", the concept of P, "unmarried men," is entailed in the concept of S, "all bachelors,"; and the sets under S and P are strictly equal with the same members of all bachelors/unmarried men. And in the case of "all dogs are mammals", the concept of P, "mammals" is contained in the concept of S, "all dogs,"; and S corresponds to the set of all dogs, which is the subset of the set of all mammals under P. **Therefore, analytic judgements presuppose a reciprocity between S and P, where the concept of S contains P, and the set under P contains S.**

From the above argument, we can see that this reciprocity between S and P in analytic judgements is not satisfied through logistic definitions of arithmetic concepts. Because firstly, the concept of P cannot be derived from S without intuitions that determine the exact objects employed in the set extensions under logicistic definitions of numbers, **violating the analytic reciprocity where the concept of P is contained in S**. And secondly, by logicistic definitions of numbers, the set C under P will always be a subset of the set C′ under S, **violating the analytic reciprocity that requires the reverse set relations where the set under P contains S**.

It is repeatedly emphasized in this section that intuition is a necessity for concepts of S and P to satisfy the containment relations of analyticity. Here, intuition is related to locating empirical objects, which according to Kant, are passively received as manifolds of sensations through the form of space. For Kant, arithmetic judgements presuppose counting, which is also intuition but in the form of time. Similarly, I also argue that Frege's attempt to deduce numbers as extensions to concepts, which renders them to be cardinalities of sets, also presupposes the idea of counting. First of all, Frege's logicistic project is predicated on Hume's principle. Meaning that the logicistic definition of numbers does not intuitively come from encountering a particular set of objects of a certain quantity. Numbers can only be mounted when someone matches objects in a set with another set in terms of the concept of cardinality, making the first step towards grasping the idea of numbers always be rewriting any statement about a number in terms of the cardinality of sets that belong to concepts. However, cardinality only arises from determining the equinumerosity of two sets, where counting is necessary**. In other words, a "counting function" of cardinality is implied under logicistic definitions of numbers, thereby leaving Frege's logicistic definitions of numbers in implicit acknowledgment of the involvement of Kantian intuition in arithmetic.**

Conclusion:

Kant defined the distinctions between analytic and synthetic judgements in terms of a concept-containment relation between subjects and predicates and the respective natures of their production, and he attributed arithmetic as synthetic a priori under these distinctions. However, he did not elaborate on numerical concepts to demonstrate further how arithmetic judgements exemplify the S-P relations of syntheticity. After Kant, Frege offered a logicistic way of defining number concepts and hoped to base arithmetic on logic which would render them analytic. However, after examination of simple arithmetic additions under Frege's logicistic definition of numbers, I find his effort in preserving arithmetic as analytic to be insufficient. I argued that Frege's definitions of the subject and predicate in arithmetic failed to fulfill the analytic reciprocity, which not only needs the concept of S to contain the concept of P, but also requires the set under P to contain the set under S. Furthermore, I argued that Frege's project entailed Kantian epistemology as it required intuition in the understanding of numbers as cardinalities. With the above two premises, I thereby conclude that Kant's assessment of arithmetic is defended against Frege's logicism that attempted to prove it otherwise.

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